

An Inverse Magnetic Problem

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In this note we describe a way to resolve an inverse magnetic problem.

1. Modeling

It is well known that an anomaly V with magnetization distribution $\overset{V}{M}(\overset{P}{r})$ is given by

$$A(\overset{P}{r}) = - \iiint_V \overset{P}{M}(\overset{P'}{r}) \cdot \nabla \frac{1}{|\overset{P}{r} - \overset{P'}{r}|} d\overset{P'}{r}$$

The scattered magnetic field caused by the anomaly V is

$$\begin{aligned} \overset{V}{B}_s(\overset{P}{r}) &= -\nabla A(\overset{P}{r}) \\ &= \nabla \iiint_V \overset{P}{M}(\overset{P'}{r}) \cdot \nabla \frac{1}{|\overset{P}{r} - \overset{P'}{r}|} d\overset{P'}{r} \end{aligned}$$

Clearly,

$$\begin{aligned} \nabla \times \overset{V}{M}(\overset{P'}{r}) &= \overset{V}{0} \\ \nabla \overset{V}{M}(\overset{P'}{r}) &= \overset{V}{0}, \end{aligned}$$

since $\overset{V}{M}(\overset{P'}{r})$ is independent of $\overset{P}{r}$.

Utilizing the formulas

$$\begin{aligned} \nabla(\overset{V}{f} \cdot \overset{P}{g}) &= \overset{V}{f} \times (\nabla \times \overset{P}{g}) + \overset{P}{g} \times (\nabla \times \overset{V}{f}) + (\overset{V}{f} \cdot \nabla) \overset{P}{g} + (\overset{P}{g} \cdot \nabla) \overset{V}{f} \\ \nabla \times \nabla \frac{1}{|\overset{P}{r} - \overset{P'}{r}|} &= \overset{P}{0} \end{aligned}$$

yields

$$\overset{P}{B}_s(\overset{P}{r}) = \nabla \iiint_V \overset{P}{M}(\overset{P'}{r}) \cdot \nabla \frac{1}{|\overset{P}{r} - \overset{P'}{r}|} d\overset{P'}{r}$$

$$\begin{aligned}
&= \iiint_V (\mathbf{M}(\mathbf{r}') \cdot \nabla) \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \\
&= \iiint_V \left(m_x \frac{\partial}{\partial x} \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} + m_y \frac{\partial}{\partial y} \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} + m_z \frac{\partial}{\partial z} \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) d\mathbf{r}'
\end{aligned}$$

where $\mathbf{M}(\mathbf{r}') = \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix}$.

Since $\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -\frac{1}{|\mathbf{r} - \mathbf{r}'|^3} ((x-x')\mathbf{i} + (y-y')\mathbf{j} + (z-z')\mathbf{k})$,

$$\begin{aligned}
\frac{\partial}{\partial x} \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} &= -\frac{\partial}{\partial x} \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} ((x-x')\mathbf{i} + (y-y')\mathbf{j} + (z-z')\mathbf{k}) \\
&= \frac{3}{2} \cdot \frac{2(x-x')}{|\mathbf{r} - \mathbf{r}'|^5} ((x-x')\mathbf{i} + (y-y')\mathbf{j} + (z-z')\mathbf{k}) - \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \mathbf{i} \\
&= \left(\frac{3(x-x')^2}{|\mathbf{r} - \mathbf{r}'|^5} - \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \right) \mathbf{i} + \frac{(x-x')(y-y')}{|\mathbf{r} - \mathbf{r}'|^5} \mathbf{j} + \frac{(x-x')(z-z')}{|\mathbf{r} - \mathbf{r}'|^5} \mathbf{k}. \quad (1)
\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial y} \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{(x-x')(y-y')}{|\mathbf{r} - \mathbf{r}'|^5} \mathbf{i} + \left(\frac{3(y-y')^2}{|\mathbf{r} - \mathbf{r}'|^5} - \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \right) \mathbf{j} + \frac{(y-y')(z-z')}{|\mathbf{r} - \mathbf{r}'|^5} \mathbf{k}, \quad (2)$$

$$\frac{\partial}{\partial z} \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{(x-x')(z-z')}{|\mathbf{r} - \mathbf{r}'|^5} \mathbf{i} + \frac{(y-y')(z-z')}{|\mathbf{r} - \mathbf{r}'|^5} \mathbf{j} + \left(\frac{3(z-z')^2}{|\mathbf{r} - \mathbf{r}'|^5} - \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \right) \mathbf{k}. \quad (3)$$

This formulation can be simplified by using matrix

$$\mathbf{B}_s(\mathbf{r}) = \iiint_V G(\mathbf{r}, \mathbf{r}') \mathbf{M}(\mathbf{r}') d\mathbf{r}',$$

where $G(\mathbf{r}, \mathbf{r}')$ is Green's tensor, a symmetric matrix $G = (g_{ij})_{3 \times 3}$ with

$$\begin{aligned}
g_{11} &= \frac{3(x-x')^2}{|\mathbf{r} - \mathbf{r}'|^5} - \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \\
g_{12} &= \frac{3(x-x')(y-y')}{|\mathbf{r} - \mathbf{r}'|^5} \\
g_{13} &= \frac{3(x-x')(z-z')}{|\mathbf{r} - \mathbf{r}'|^5}
\end{aligned}$$

$$g_{21} = \frac{3(x-x')(y-y')}{|\mathbf{r}-\mathbf{r}'|^5}$$

$$g_{22} = \frac{3(y-y')^2}{|\mathbf{r}-\mathbf{r}'|^5} - \frac{1}{|\mathbf{r}-\mathbf{r}'|^3}$$

$$g_{23} = \frac{3(y-y')(z-z')}{|\mathbf{r}-\mathbf{r}'|^5}$$

$$g_{31} = \frac{3(x-x')(z-z')}{|\mathbf{r}-\mathbf{r}'|^5}$$

$$g_{32} = \frac{3(y-y')(z-z')}{|\mathbf{r}-\mathbf{r}'|^5}$$

$$g_{33} = \frac{3(z-z')^2}{|\mathbf{r}-\mathbf{r}'|^5} - \frac{1}{|\mathbf{r}-\mathbf{r}'|^3},$$

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}.$$

Therefore the total field is

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_e + \mathbf{B}_s(\mathbf{r}) = \mathbf{B}_e + \iiint_V G(\mathbf{r}, \mathbf{r}') \mathbf{M}(\mathbf{r}') d\mathbf{r}',$$

where $\mathbf{B}_e(\mathbf{r})$ represents the earth's magnetic field.

We now break the anomaly into M cells and assume that there exists exactly one cell having magnetization. The task is to identify the location of this particular cell, cell j say, and compute its magnetization (amplitude as well as direction) as well. The information we have is the measured total magnetic field:

$$|\mathbf{B}(\mathbf{r}_1)|, |\mathbf{B}(\mathbf{r}_2)|, \dots, |\mathbf{B}(\mathbf{r}_N)|.$$

The problem can be formulated as

$$|\mathbf{B}(\mathbf{r}_i)| = |\mathbf{B}_e + G(\mathbf{r}_i, \mathbf{r}_j) \mathbf{M}_j \Delta V_j|, \quad i = 1, 2, \dots, N,$$

where \mathbf{M}_j stands for the magnetization of cell j , and \mathbf{r}_j specifies the coordinates of the center of cell j .

We introduce the notation

$$\overset{\rho}{B}(f_i) = \begin{pmatrix} b_1^i \\ b_2^i \\ b_3^i \end{pmatrix}$$

$$\overset{\rho}{B}_e = \begin{pmatrix} b_1^e \\ b_2^e \\ b_3^e \end{pmatrix}$$

$$\overset{\rho}{M}_j = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

$$G(\overset{\rho}{f}_i, \overset{\rho}{f}_j) \Delta V_j = (a_{kl})_{3 \times 3}.$$

Note that our aim is to determine the vector $\overset{\rho}{M}_j = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$.

Using the above notation, we have N equations, each having the form

$$\begin{aligned} m_1^2 \sum_{i=1}^3 a_{i1}^2 + m_2^2 \sum_{i=1}^3 a_{i2}^2 + m_3^2 \sum_{i=1}^3 a_{i3}^2 + 2m_1 m_2 \sum_{i=1}^3 a_{i1} a_{i2} + 2m_1 m_3 \sum_{i=1}^3 a_{i1} a_{i3} + 2m_2 m_3 \sum_{i=1}^3 a_{i2} a_{i3} \\ + 2m_1 \sum_{i=1}^3 a_{i1} u_i + 2m_2 \sum_{i=1}^3 a_{i2} u_i + 2m_3 \sum_{i=1}^3 a_{i3} u_i = \sum_{k=1}^3 b_k^i - \sum_{k=1}^3 b_k^e, \quad i = 1, 2, \dots, N. \end{aligned}$$

Solving this equation system for m_1, m_2, m_3 (by means of the least squares) gives the desired results.

Based on the testing we conducted, the magnitude of the desired magnetization vector as well as its direction can be localized accurately if the volume of the body is presumably known. However, if we don't know exactly what the volume is, the results are less accurate. To resolve this issue, we would take the information about derivatives of B_{total} into consideration. To this end, the following computation is carried out. It follows from (1) that

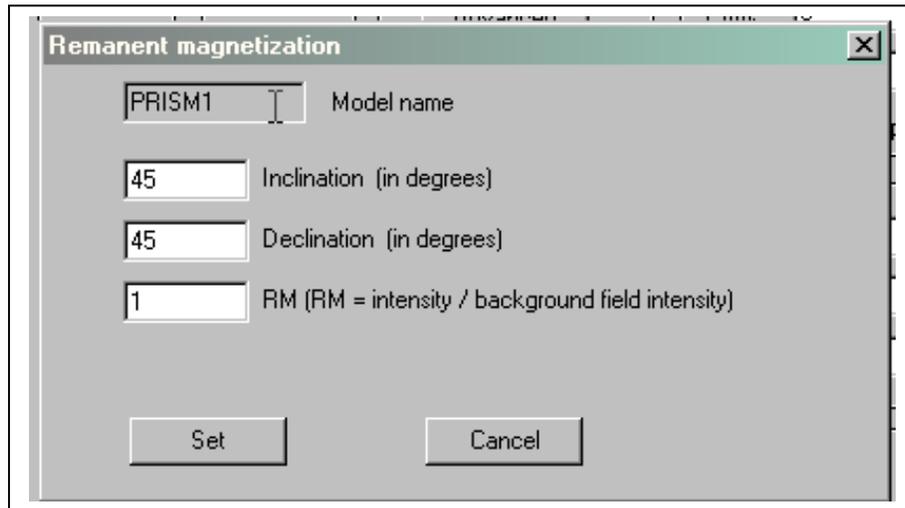
$$\begin{aligned}
\frac{\partial^2}{\partial x^2} \nabla \frac{1}{|F-F'|} &= \frac{\partial}{\partial x} \left[\left(\frac{3(x-x')^2}{|F-F'|^5} - \frac{1}{|F-F'|^3} \right) i + \frac{(x-x')(y-y')}{|F-F'|^5} j + \frac{(x-x')(z-z')}{|F-F'|^5} k \right] \\
&= \left(-\frac{5}{2} \cdot \frac{3(x-x')^2 \cdot 2(x-x')}{|F-F'|^7} + \frac{6(x-x')}{|F-F'|^5} + \frac{3 \cdot 2(x-x')}{2|F-F'|^5} \right) i + \left(-\frac{5}{2} \cdot \frac{(x-x')(y-y') \cdot 2(x-x')}{|F-F'|^7} + \frac{(y-y')}{|F-F'|^5} \right) j \\
&\quad + \left(-\frac{5}{2} \cdot \frac{(x-x')(z-z') \cdot 2(x-x')}{|F-F'|^7} + \frac{(z-z')}{|F-F'|^5} \right) k \\
&= \left(-\frac{15(x-x')^3}{|F-F'|^7} + \frac{9(x-x')}{|F-F'|^5} \right) i + \left(-\frac{5(x-x')^2(y-y')}{|F-F'|^7} + \frac{(y-y')}{|F-F'|^5} \right) j \\
&\quad + \left(-\frac{5(x-x')^2(z-z')}{|F-F'|^7} + \frac{(z-z')}{|F-F'|^5} \right) k.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial x \partial y} \nabla \frac{1}{|F-F'|} &= \frac{\partial}{\partial y} \left[\left(\frac{3(x-x')^2}{|F-F'|^5} - \frac{1}{|F-F'|^3} \right) i + \frac{(x-x')(y-y')}{|F-F'|^5} j + \frac{(x-x')(z-z')}{|F-F'|^5} k \right] \\
&= \left(-\frac{5}{2} \cdot \frac{3(x-x')^2 \cdot 2(y-y')}{|F-F'|^7} + \frac{3 \cdot 2(y-y')}{2|F-F'|^5} \right) i + \left(-\frac{5}{2} \cdot \frac{(x-x')(y-y') \cdot 2(y-y')}{|F-F'|^7} + \frac{x-x'}{|F-F'|^5} \right) j \\
&\quad + \left(-\frac{5}{2} \cdot \frac{(x-x')(z-z') \cdot 2(z-z')}{|F-F'|^7} \right) k \\
&= \left(-\frac{5}{2} \cdot \frac{3(x-x')^2 \cdot 2(y-y')}{|F-F'|^7} - \frac{3 \cdot 2(y-y')}{2|F-F'|^5} \right) i + \left(-\frac{5}{2} \cdot \frac{(x-x')(y-y') \cdot 2(y-y')}{|F-F'|^7} + \frac{x-x'}{|F-F'|^5} \right) j \\
&\quad + \left(-\frac{5}{2} \cdot \frac{(x-x')(z-z') \cdot 2(z-z')}{|F-F'|^7} \right) k
\end{aligned}$$

2. Testing

The testing executable is located at Petros-12\E:\users\rjia\inverse_mag\release

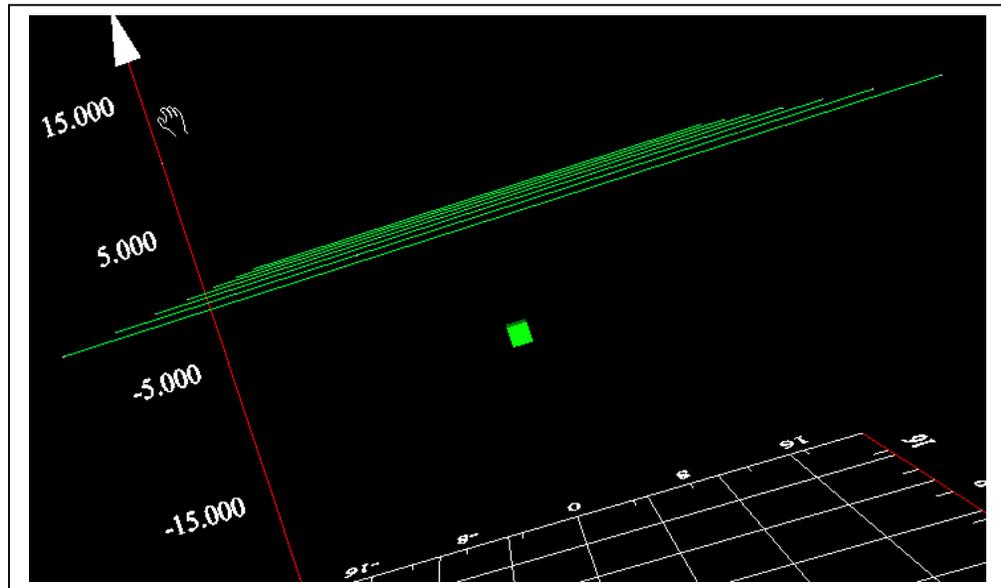
The first Test: remanent body



The screenshot shows a dialog box titled "Remanent magnetization" with a close button (X) in the top right corner. The dialog contains the following fields and labels:

- Model name: PRISM1
- Inclination (in degrees): 45
- Declination (in degrees): 45
- RM (RM = intensity / background field intensity): 1

At the bottom of the dialog, there are two buttons: "Set" and "Cancel".



Model Properties

Dimensions		Center/Top	
Strike Length	<input type="text" value="1"/>	North	<input type="text" value="0"/>
Thickness	<input type="text" value="1"/>	Up	<input type="text" value="-5.5"/>
Dip Extent	<input type="text" value="1"/>	East	<input type="text" value="0"/>
Material Properties		Geological Angles	
Conductivity	<input type="text" value="0.01"/>	Strike	<input type="text" value="0"/>
Susceptibility	<input type="text" value="0"/>	Dip	<input type="text" value="90"/>
Permittivity	<input type="text" value="1"/>	Plunge	<input type="text" value="0"/>
Resistivity	<input type="text" value="100"/>	Number Of Sample Pts <input type="text" value="1"/>	
<input type="button" value="Apply"/>	<input type="button" value="Color"/>	Name <input type="text" value="PRISM1"/>	
<input type="button" value="Undo"/>	<input type="button" value="Close"/>		

Top Center

Profile | Waveform | Tx-Rx | Prisms/Plates/Polyhedra | Layers | Output | EMSphere

Profile #	Num. Loc.
1 LINE1	7
2 LINE2	7
3 LINE3	7
4 LINE4	7
5 LINE5	7
6 LINE6	7
7 LINE7	7

Total number of profiles: 7
 Total number of stations: 49
 Profile: 1
 Profile Name: LINE1
 Station #: 7

Reorder Profiles

Station	Profile #	X	Y	Z
1	1	-1.500000e+001	-1.500000e+001	1.000000e+000
2	1	-1.000000e+001	-1.500000e+001	1.000000e+000
3	1	-5.000000e+000	-1.500000e+001	1.000000e+000
4	1	0.000000e+000	-1.500000e+001	1.000000e+000
5	1	5.000000e+000	-1.500000e+001	1.000000e+000
6	1	1.000000e+001	-1.500000e+001	1.000000e+000
7	1	1.500000e+001	-1.500000e+001	1.000000e+000

Responses Channels

Modify Profile

Delete Every: 2 Apply
 Shift Z: 0 Apply
 All Profiles Current Profile

Join Profiles: 0 Apply
 Split Current Profile Before Selected Station: Apply
 Change Name: LINE1 Apply

Add Single Station

X: 15
 Y: -15
 Z: 1
 <-- Add to the Profile
 Replace Insert

Generate Stations with Constant Step

First Station: X: -15 Y: -15 Z: 1
 Last Station: X: 15 Y: -15 Z: 1
 Station Increment: 5 Num. of Stations: 7

New Profile
 Replace Profile
 Unit: Meter Feet

Import Profile
 Profiles On Topography
 Retrieve/Restore Data

Profiles information

The testing results

Inverse_Mag

Gridding Information

XStart: -2 XEnd: 2
 YMin: -2 YMax: 2
 ZMin: -10 ZMax: 0
 Delta X: 0.5 Delta Y: 0.5 Delta Z: 0.5

Earth Field System (Background)

Inclination (degrees): 75
 Declination (degrees): 20
 Intensity (nT): 52500

Input File: E:\Users\vija\testing\alter.xyz Browse
 Output File: E:\Users\vija\testing\vija.txt Browse

OK Read Run

Simulated Results

Source Cell Coordinates
 X: -0.25 Y: -0.25 Z: -5.75

Source Magnetization
 Mx: 25961.402031836
 My: 25610.84917573
 Mz: -40196.12257313

Relative Magnitude over Earth: 1.033785419802

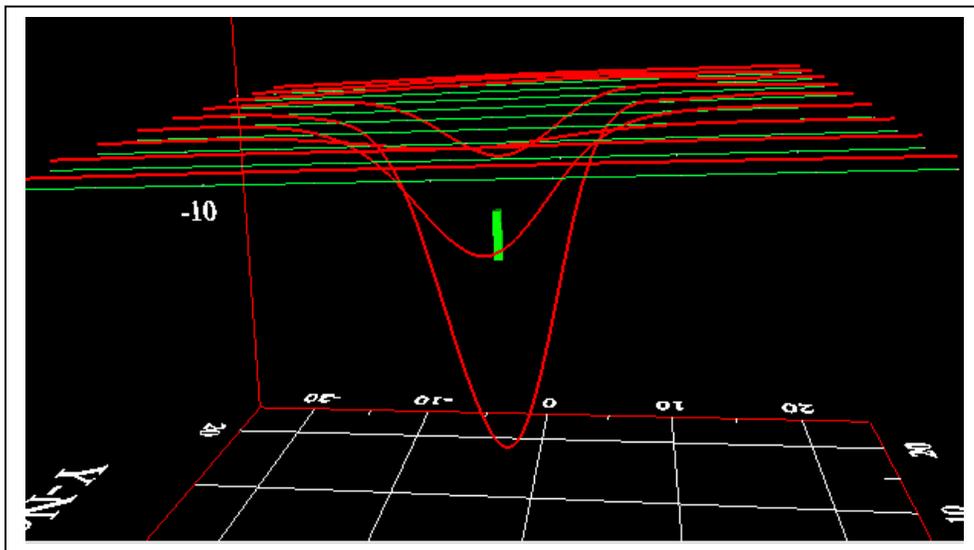
Note in the output,
$$\begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} = \overset{\rho}{M}_j \Delta V_j = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \Delta V_j.$$

In the output file (jia.txt as specified in the above), the first three numbers in each row indicate the coordinates of the center of a cell, followed by error

$$\mu = \frac{1}{N} \sum_{i=1}^N \left\| \overset{\rho}{B}(r_i) - \left| \overset{\rho}{B}_e + G(r_i, r_j) \overset{\rho}{M}_j \Delta V_j \right| \right\|$$

and deviation.

The second Test: (prism not remanent)



Model Properties

Dimensions

Strike Length

Thickness

Dip Extent

Center/Top

North

Up

East

Top Center

Material Properties

Conductivity

Susceptibility

Permittivity

Resistivity

Geological Angles

Strike

Dip

Plunge

Number Of Sample Pts

Name

Inverse_Mag

Gridding Information

XStart

XEnd

YMin

YMax

ZMin

ZMax

Delta X

Delta Y

Delta Z

Earth Field System (Background)

Inclination (degrees)

Declination (degrees)

Intensity (nT)

Input File

Output File

Simulated Results

Source Cell Coordinates

X

Y

Z

Source Magnetization

Mx

My

Mz

Relative Magnitude over Earth

Profiles | Waveform | Tx/Rx | Prisms/Plates/Polyhedra | Layers | Output | EMSphere

Profile #	Num. Loc.
1 LINE1	9
2 LINE2	9
3 LINE3	9
4 LINE4	9
5 LINE5	9
6 LINE6	9
7 LINE7	9
8 LINE8	9

Reorder Profiles

Total number of profiles: 9
 Total number of stations: 81
 Profile: 1
 Profile Name: LINE1
 Station #: 9

Station	Profile #	X	Y	Z
1	1	-2.000000e+001	-2.000000e+001	5.000000e-001
2	1	-1.500000e+001	-2.000000e+001	5.000000e-001
3	1	-1.000000e+001	-2.000000e+001	5.000000e-001
4	1	-5.000000e+000	-2.000000e+001	5.000000e-001
5	1	0.000000e+000	-2.000000e+001	5.000000e-001
6	1	5.000000e+000	-2.000000e+001	5.000000e-001
7	1	1.000000e+001	-2.000000e+001	5.000000e-001
8	1	1.500000e+001	-2.000000e+001	5.000000e-001
9	1	2.000000e+001	-2.000000e+001	5.000000e-001

Responses Channels

Modify Profile

Delete Every: 2 Apply
 Shift Z: 0 Apply
 All Profiles Current Profile

Join Profiles: 0 Apply
 Split Current Profile Before Selected Station: Apply
 Change Name: LINE1 Apply

Add Single Station

X: 20
 Y: -20
 Z: 0.5
 <-- Add to the Profile
 Replace Insert

Generate Stations with Constant Step

First Station: X: -20 Y: -20 Z: 0.5
 Last Station: X: 20 Y: -20 Z: 0.5
 Station Increment: 5 Num. of Stations: 9
 New Profile
 Replace Profile
 Unit: Meter Feet

Import Profile
 Profiles On Topography
 Retrieve/Restore Data

Profiles Information