An Inverse Magnetic Problem

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In this note we describe a way to resolve an inverse magnetic problem.

1. **Modeling**

It is well known that an anomaly V with magnetization distribution $\vec{M}(\vec{r})$ is given by

$$A(\vec{r}) = -\iiint_V \vec{M}(\vec{r}') \bullet \nabla \frac{1}{|\vec{r}-\vec{r}'|} d\vec{r}'$$

The scattered magnetic filed caused by the anomaly V is

$$\begin{aligned} \vec{B}_s(\vec{r}) &= -\nabla A(\vec{r}) \\ &= \nabla \iiint_V \vec{M}(\vec{r}') \bullet \nabla \frac{1}{|\vec{r}-\vec{r}'|} d\vec{r}' \end{aligned}$$

Clearly,

$$\nabla \times \vec{M}(\vec{r}') = \vec{0}$$
$$\nabla \vec{M}(\vec{r}') = \vec{0} \ ,$$

since $\vec{M}(\vec{r}')$ is independent of $\vec{r}$.

Utilizing the formulas

$$\nabla(\vec{f} \bullet \vec{g}) = \vec{f} \times (\nabla \times \vec{g}) + \vec{g} \times (\nabla \times \vec{f}) + (\vec{f} \bullet \nabla)\vec{g} + (\vec{g} \bullet \nabla)\vec{f}$$
$$\nabla \times \nabla \frac{1}{|\vec{r}-\vec{r}'|} = \vec{0}$$

yields

$$\vec{B}_s(\vec{r}) = \nabla \iiint_V \vec{M}(\vec{r}') \bullet \nabla \frac{1}{|\vec{r}-\vec{r}'|} d\vec{r}'$$

$$= \iiint_V (\vec{M}(\vec{r}') \bullet \nabla)\nabla \frac{1}{|\vec{r}-\vec{r}'|} d\vec{r}'$$

$$= \iiint_V (m_x \frac{\partial}{\partial x}\nabla \frac{1}{|\vec{r}-\vec{r}'|} + m_y \frac{\partial}{\partial y}\nabla \frac{1}{|\vec{r}-\vec{r}'|} + m_z \frac{\partial}{\partial z}\nabla \frac{1}{|\vec{r}-\vec{r}'|}) d\vec{r}'$$

where $\vec{M}(\vec{r}') = \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix}$.

Since $\nabla \frac{1}{|\vec{r}-\vec{r}'|} = -\frac{1}{|\vec{r}-\vec{r}'|^3}((x-x')\vec{i} + (y-y')\vec{j} + (z-z')\vec{k})$,

$$\frac{\partial}{\partial x}\nabla \frac{1}{|\vec{r}-\vec{r}'|} = -\frac{\partial}{\partial x}\frac{1}{|\vec{r}-\vec{r}'|^3}((x-x')\vec{i} + (y-y')\vec{j} + (z-z')\vec{k})$$

$$= \frac{3}{2}\cdot\frac{2(x-x')}{|\vec{r}-\vec{r}'|^5}((x-x')\vec{i} + (y-y')\vec{j} + (z-z')\vec{k}) - \frac{1}{|\vec{r}-\vec{r}'|^3}\vec{i}$$

$$= \left(\frac{3(x-x')^2}{|\vec{r}-\vec{r}'|^5} - \frac{1}{|\vec{r}-\vec{r}'|^3}\right)\vec{i} + \frac{(x-x')(y-y')}{|\vec{r}-\vec{r}'|^5}\vec{j} + \frac{(x-x')(z-z')}{|\vec{r}-\vec{r}'|^5}\vec{k}. \quad (1)$$

Similarly,

$$\frac{\partial}{\partial y}\nabla \frac{1}{|\vec{r}-\vec{r}'|} = \frac{(x-x')(y-y')}{|\vec{r}-\vec{r}'|^5}\vec{i} + \left(\frac{3(y-y')^2}{|\vec{r}-\vec{r}'|^5} - \frac{1}{|\vec{r}-\vec{r}'|^3}\right)\vec{j} + \frac{(y-y')(z-z')}{|\vec{r}-\vec{r}'|^5}\vec{k}, \quad (2)$$

$$\frac{\partial}{\partial z}\nabla \frac{1}{|\vec{r}-\vec{r}'|} = \frac{(x-x')(z-z')}{|\vec{r}-\vec{r}'|^5}\vec{i} + \frac{(y-y')(z-z')}{|\vec{r}-\vec{r}'|^5}\vec{j} + \left(\frac{3(z-z')^2}{|\vec{r}-\vec{r}'|^5} - \frac{1}{|\vec{r}-\vec{r}'|^3}\right)\vec{k}. \quad (3)$$

This formulation can be simplified by using matrix

$$\vec{B}_s(\vec{r}) = \iiint_V G(\vec{r},\vec{r}')\vec{M}(\vec{r}')d\vec{r}',$$

where $G(\vec{r},\vec{r}')$ is Green's tensor, a symmetric matrix $G = (g_{ij})_{3\times 3}$ with

$$g_{11} = \frac{3(x-x')^2}{|\vec{r}-\vec{r}'|^5} - \frac{1}{|\vec{r}-\vec{r}'|^3}$$

$$g_{12} = \frac{3(x-x')(y-y')}{|\vec{r}-\vec{r}'|^5}$$

$$g_{13} = \frac{3(x-x')(z-z')}{|\vec{r}-\vec{r}'|^5}$$

$$g_{21} = \frac{3(x-x')(y-y')}{|\vec{r}-\vec{r}'|^5}$$

$$g_{22} = \frac{3(y-y')^2}{|\vec{r}-\vec{r}'|^5} - \frac{1}{|\vec{r}-\vec{r}'|^3}$$

$$g_{23} = \frac{3(y-y')(z-z')}{|\vec{r}-\vec{r}'|^5}$$

$$g_{31} = \frac{3(x-x')(z-z')}{|\vec{r}-\vec{r}'|^5}$$

$$g_{32} = \frac{3(y-y')(z-z')}{|\vec{r}-\vec{r}'|^5}$$

$$g_{33} = \frac{3(z-z')^2}{|\vec{r}-\vec{r}'|^5} - \frac{1}{|\vec{r}-\vec{r}'|^3},$$

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \vec{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}.$$

Therefore the total field is

$$\vec{B}(\vec{r}) = \vec{B}_e + \vec{B}_s(\vec{r}) = \vec{B}_e + \iiint_V G(\vec{r},\vec{r}')\vec{M}(\vec{r}')d\vec{r}',$$

where $\vec{B}_e(\vec{r})$ represents the earth's magnetic field.

We now break the anomaly into M cells and assume that there exists exactly one cell having magnetization. The task is to identify the location of this particular cell, cell j say, and compute its magnetization (amplitude as well as direction) as well. The information we have is the measured total magnetic field:

$$\left|\vec{B}(\vec{r}_1)\right|, \left|\vec{B}(\vec{r}_2)\right|, \ldots, \left|\vec{B}(\vec{r}_N)\right|.$$

The problem can be formulated as

$$\left|\vec{B}(\vec{r}_i)\right| = \left|\vec{B}_e + G(\vec{r}_i,\vec{r}_j)\vec{M}_j \Delta V_j\right|, \quad i = 1, 2, \ldots, N,$$

where $\vec{M}_j$ stands for the magnetization of cell j, and $\vec{r}_j$ specifies the coordinates of the center of cell j.

We introduce the notation

$$\vec{B}(\vec{r}_i) = \begin{pmatrix} b_1^i \\ b_2^i \\ b_3^i \end{pmatrix}$$

$$\vec{B}_e = \begin{pmatrix} b_1^e \\ b_2^e \\ b_3^e \end{pmatrix}$$

$$\vec{M}_j = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

$$G(\vec{r}_i, \vec{r}_j)\Delta V_j = (a_{kl})_{3\times 3}\,.$$

Note that our aim is to determine the vector $\vec{M}_j = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$.

Using the above notation, we have N equations, each having the form

$$m_1^2 \sum_{i=1}^{3} a_{i1}^2 + m_2^2 \sum_{i=1}^{3} a_{i2}^2 + m_3^2 \sum_{i=1}^{3} a_{i3}^2 + 2m_1m_2 \sum_{i=1}^{3} a_{i1}a_{i2} + 2m_1m_3 \sum_{i=1}^{3} a_{i1}a_{i3} + 2m_{21}m_3 \sum_{i=1}^{3} a_{i2}a_{i3}$$

$$+ 2m_1 \sum_{i=1}^{3} a_{i1}u_i + 2m_2 \sum_{i=1}^{3} a_{i2}u_i + 2m_3 \sum_{i=1}^{3} a_{i3}u_i = \sum_{k=1}^{3} b_k^i - \sum_{k=1}^{3} b_k^e\,,\ i = 1, 2, \ldots, N.$$

Solving this equation system for m_1, m_2, m_3 (by means of the least squares) gives the desired results.

Based on the testing we conducted, the magnitude of the desired magnetization vector as well as its direction can be localized accurately if the volume of the body is presumably known. However, if we don't know exactly what the volume is, the results are less accurate. To resolve this issue, we would take the information about derivatives of Btotal into consideration. To this end, the following computation is carried out. It follows from (1) that

$$\frac{\partial^2}{\partial x^2}\nabla\frac{1}{|\vec{r}-\vec{r}'|}=\frac{\partial}{\partial x}\left[\left(\frac{3(x-x')^2}{|\vec{r}-\vec{r}'|^5}-\frac{1}{|\vec{r}-\vec{r}'|^3}\right)\vec{i}+\frac{(x-x')(y-y')}{|\vec{r}-\vec{r}'|^5}\vec{j}+\frac{(x-x')(z-z')}{|\vec{r}-\vec{r}'|^5}\vec{k}\right]$$

$$=\left(-\frac{5}{2}\cdot\frac{3(x-x')^2\cdot 2(x-x')}{|\vec{r}-\vec{r}'|^7}+\frac{6(x-x')}{|\vec{r}-\vec{r}'|^5}+\frac{3\cdot 2(x-x')}{2|\vec{r}-\vec{r}'|^5}\right)\vec{i}+\left(-\frac{5}{2}\cdot\frac{(x-x')(y-y')2(x-x')}{|\vec{r}-\vec{r}'|^7}+\frac{(y-y')}{|\vec{r}-\vec{r}'|^5}\right)\vec{j}$$

$$+\left(-\frac{5}{2}\cdot\frac{(x-x')(z-z')2(x-x')}{|\vec{r}-\vec{r}'|^7}+\frac{(z-z')}{|\vec{r}-\vec{r}'|^5}\right)\vec{k}$$

$$=\left(-\frac{15(x-x')^3}{|\vec{r}-\vec{r}'|^7}+\frac{9(x-x')}{|\vec{r}-\vec{r}'|^5}\right)\vec{i}+\left(-\frac{5(x-x')^2(y-y')}{|\vec{r}-\vec{r}'|^7}+\frac{(y-y')}{|\vec{r}-\vec{r}'|^5}\right)\vec{j}$$

$$+\left(-\frac{5(x-x')^2(z-z')}{|\vec{r}-\vec{r}'|^7}+\frac{(z-z')}{|\vec{r}-\vec{r}'|^5}\right)\vec{k}\,.$$

$$\frac{\partial}{\partial x\partial y}\nabla\frac{1}{|\vec{r}-\vec{r}'|}=\frac{\partial}{\partial y}\left[\left(\frac{3(x-x')^2}{|\vec{r}-\vec{r}'|^5}-\frac{1}{|\vec{r}-\vec{r}'|^3}\right)\vec{i}+\frac{(x-x')(y-y')}{|\vec{r}-\vec{r}'|^5}\vec{j}+\frac{(x-x')(z-z')}{|\vec{r}-\vec{r}'|^5}\vec{k}\right]$$

$$=\left(-\frac{5}{2}\cdot\frac{3(x-x')^2\cdot 2(y-y')}{|\vec{r}-\vec{r}'|^7}+\frac{3\cdot 2(y-y')}{2|\vec{r}-\vec{r}'|^5}\right)\vec{i}+\left(-\frac{5}{2}\cdot\frac{(x-x')(y-y')\cdot 2(y-y')}{|\vec{r}-\vec{r}'|^7}+\frac{x-x'}{|\vec{r}-\vec{r}'|^5}\right)\vec{j}$$

$$+\left(-\frac{5}{2}\cdot\frac{(x-x')(z-z')\cdot 2(z-z')}{|\vec{r}-\vec{r}'|^7}\right)\vec{k}$$

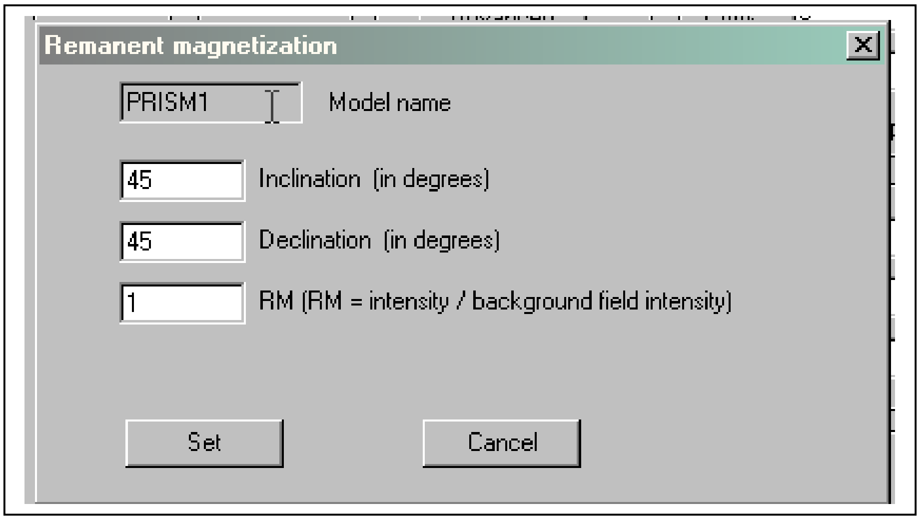
$$=\left(-\frac{5}{2}\cdot\frac{3(x-x')^2\cdot 2(y-y')}{|\vec{r}-\vec{r}'|^7}-\frac{3\cdot 2(y-y')}{2|\vec{r}-\vec{r}'|^5}\right)\vec{i}+\left(-\frac{5}{2}\cdot\frac{(x-x')(y-y')\cdot 2(y-y')}{|\vec{r}-\vec{r}'|^7}+\frac{x-x'}{|\vec{r}-\vec{r}'|^5}\right)\vec{j}$$

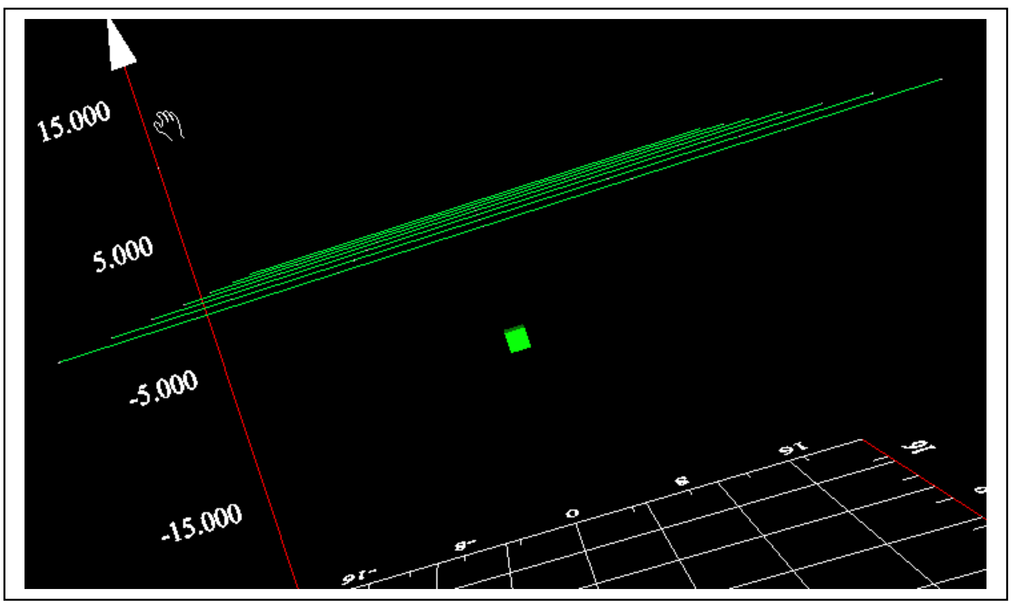
$$+\left(-\frac{5}{2}\cdot\frac{(x-x')(z-z')\cdot 2(z-z')}{|\vec{r}-\vec{r}'|^7}\right)\vec{k}$$

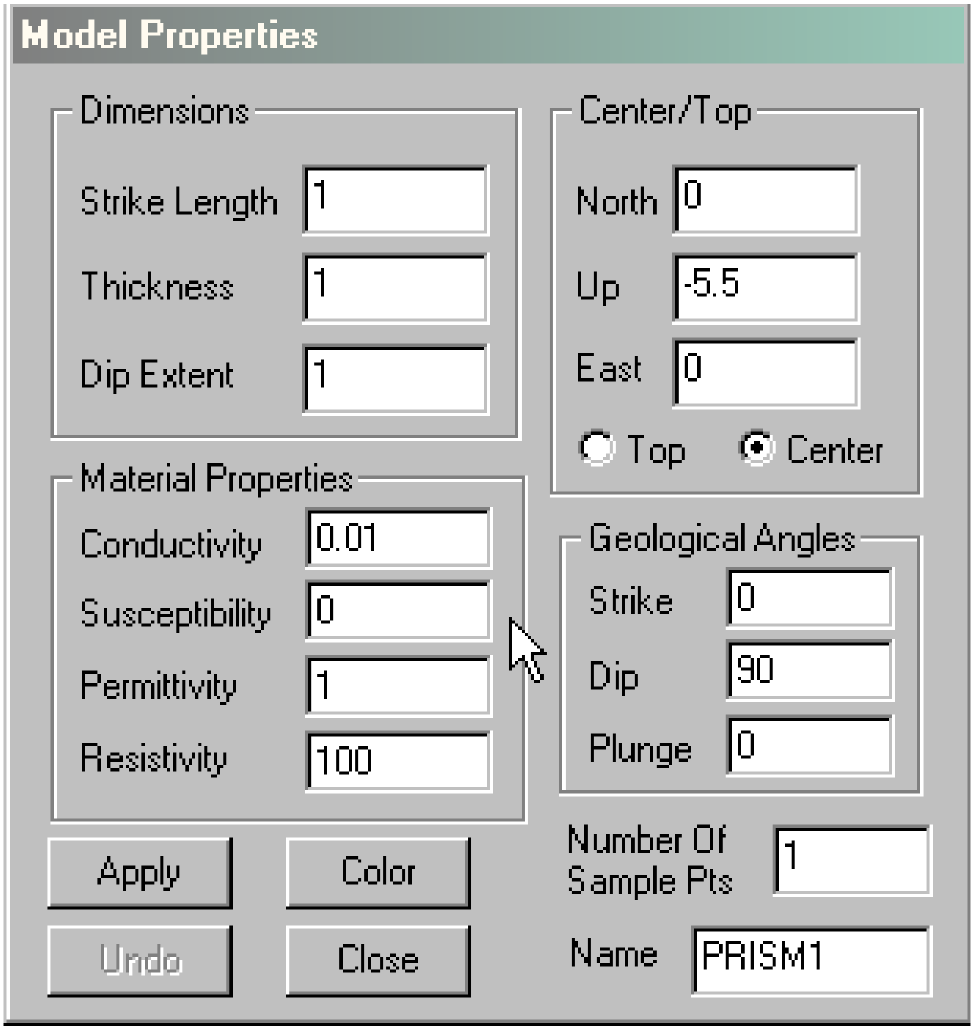
2. Testing

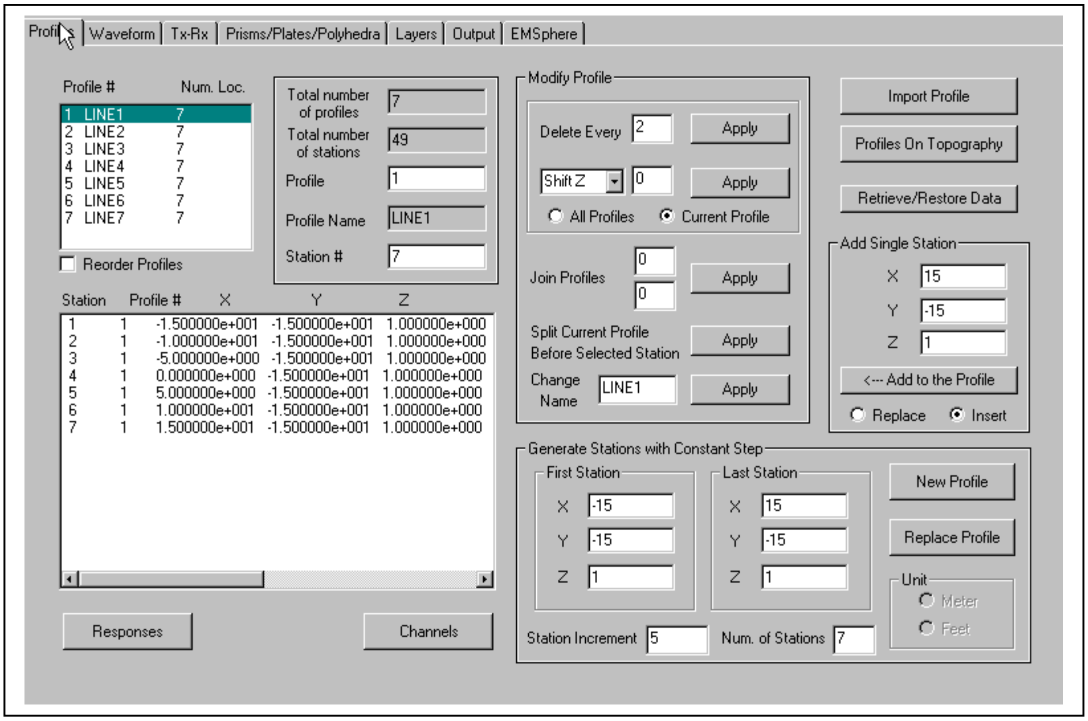
The testing executable is located at Petros-12\E:\users\rjia\inverse_mag\release

The first Test: remanent body



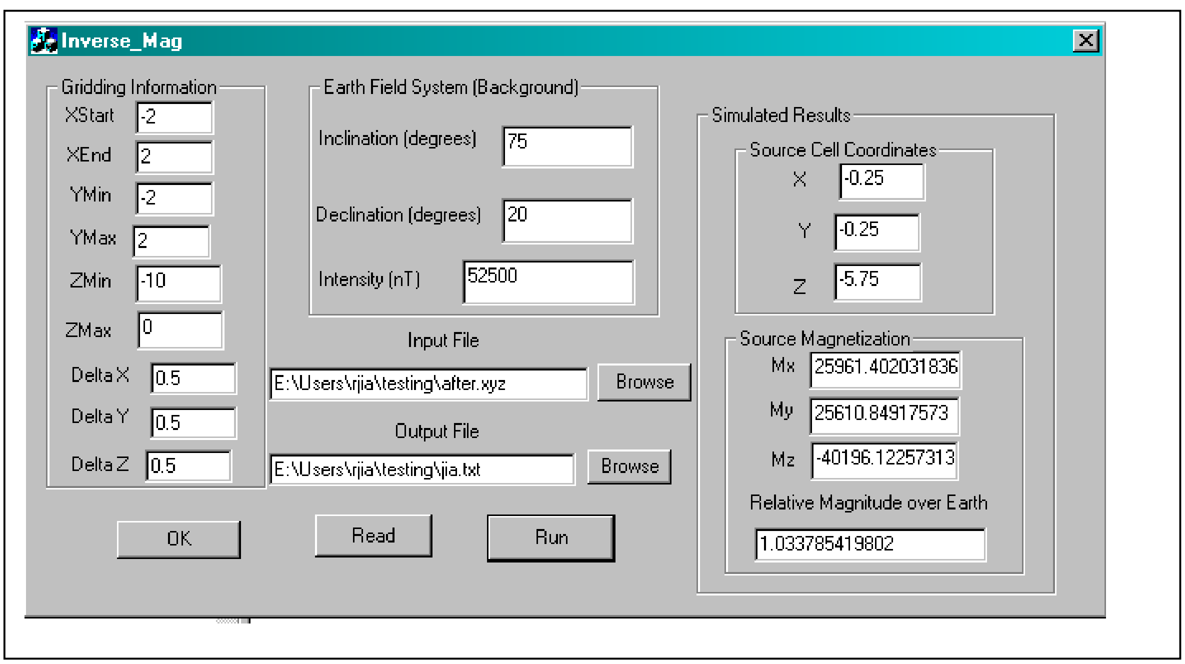



Model Properties
Dimensions
Strike Length
1
Thickness
1
Dip Extent
1
Center/Top
North
0
Up
-5.5
East
0
Top
Center
Material Properties
Conductivity
0.01
Susceptibility
0
Permittivity
1
Resistivity
100
Geological Angles
Strike
0
Dip
90
Plunge
0
Number Of Sample Pts
1
Apply
Color
Undo
Close
Name
PRISM1



Profiles information

The testing results



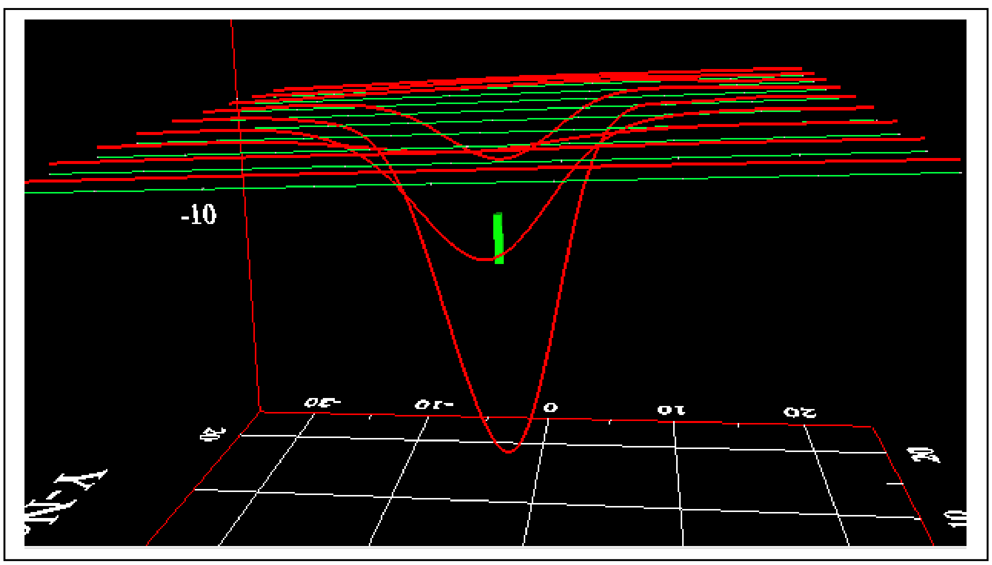
Note in the output, $\begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} = \vec{M}_j \Delta V_j = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \Delta V_j$.

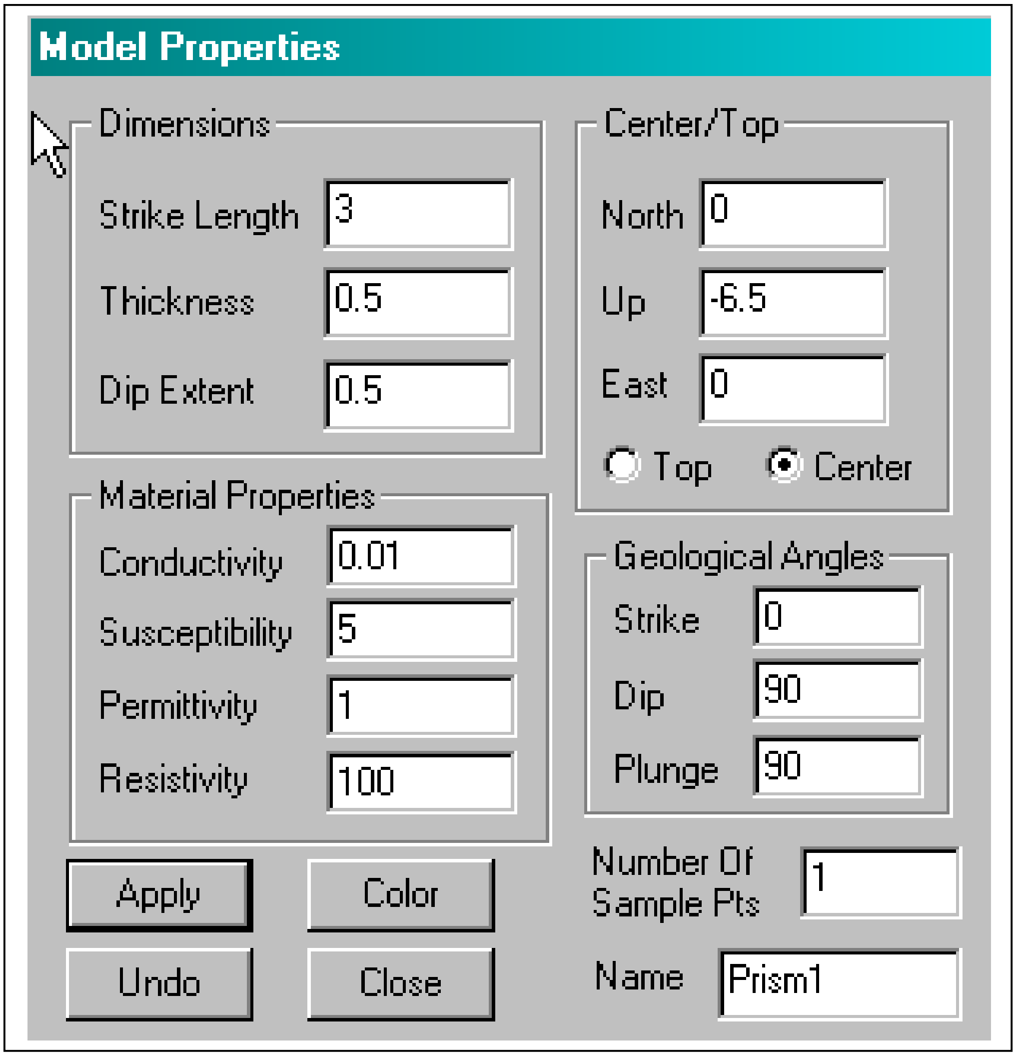
In the output file (jia.txt as specified in the above), the first three numbers in each row indicate the coordinates of the center of a cell, followed by error

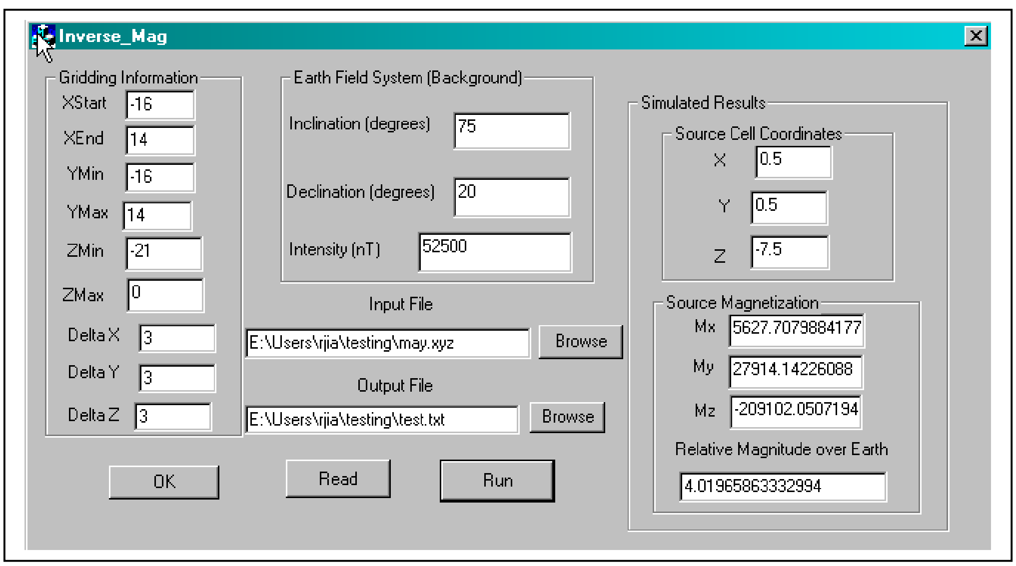
$$\mu = \frac{1}{N} \sum_{i=1}^{N} \left\| \left| \vec{B}(\vec{r}_i) \right| - \left| \vec{B}_e + G(\vec{r}_i, \vec{r}_j) \vec{M}_j \Delta V_j \right| \right\|$$

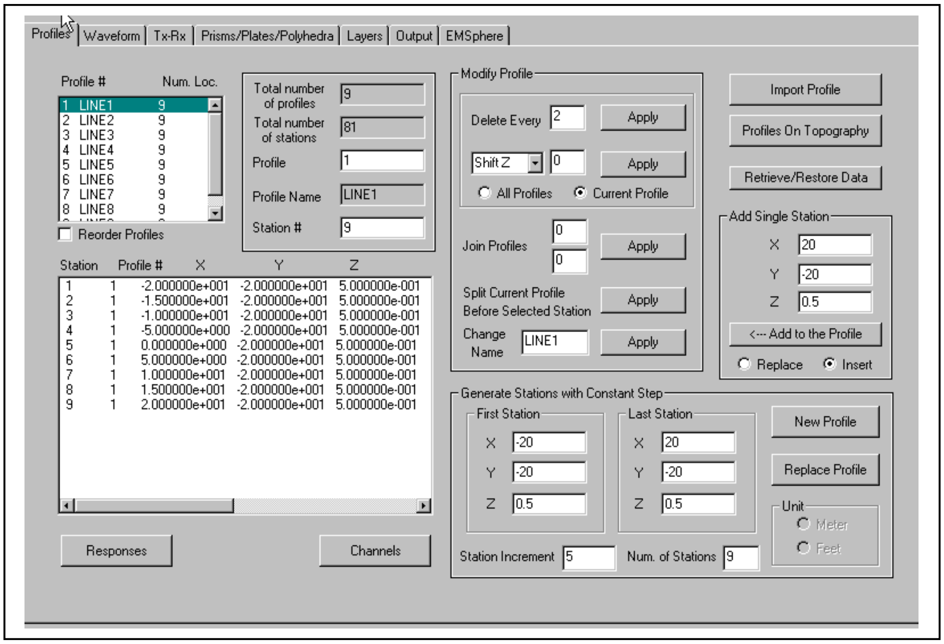
and deviation.

The second Test: (prism not remanent)




Model Properties
Dimensions
Strike Length
3
Thickness
0.5
Dip Extent
0.5
Center/Top
North
0
Up
-6.5
East
0
Top
Center
Material Properties
Conductivity
0.01
Susceptibility
5
Permittivity
1
Resistivity
100
Geological Angles
Strike
0
Dip
90
Plunge
90
Apply
Color
Number Of Sample Pts
1
Undo
Close
Name
Prism1


Inverse_Mag
Gridding Information
XStart -16
XEnd 14
YMin -16
YMax 14
ZMin -21
ZMax 0
Delta X 3
Delta Y 3
Delta Z 3
Earth Field System (Background)
Inclination (degrees) 75
Declination (degrees) 20
Intensity (nT) 52500
Input File
E:\Users\rjia\testing\may.xyz
Browse
Output File
E:\Users\rjia\testing\test.txt
Browse
Simulated Results
Source Cell Coordinates
X 0.5
Y 0.5
Z -7.5
Source Magnetization
Mx 5627.7079884177
My 27914.14226088
Mz -209102.0507194
Relative Magnitude over Earth
4.01965863332994
OK
Read
Run



Profiles Information